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CIRCULAR ARCH WITH NO INSTABILITY

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Abstract—The compressed circular arch with no instability is reported. Both ends of the arch are guided and one end loaded by a normal concentrated force. The exact solution of the non-linear governing equations and stability analysis by the dynamic method show that the primary path remains stable as load tends to infinity. There is an infinite number of bifurcation points, not located on the primary path, and two branches of secondary path (one stable and one unstable) emanate from the first bifurcation point. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The stability of circular arches under a concentrated load at the crown was analyzed by Huddleston (1968), DaDeppo and Schmidt (1969) for hinged arches, DaDeppo and Schmidt (1975) for a clamped-hinged arch, and Schmidt and DaDeppo (1970) considered an eccentric load. The load–displacement curves in these works displayed on the primary path a maximum, at which point the arches lost stability by the snap-through.

In the present paper we demonstrate the simple model, subjected to compressive stresses, the primary path of which remains stable for any value of the applied concentrated force. On the load–rotation curve of this model there is an infinite number of bifurcation points, not located on the primary path, and two branches of secondary path (one stable and one unstable) emanate from the first bifurcation point.

The model under consideration is a circular arch with both ends guided and one end loaded by a normal concentrated force. The primary path for this arch is the monotone increasing function of the rotation and it remains stable as load tends to infinity. Since the stability is analyzed by the Lyapunov's method of first approximation (the dynamic equations are linerized with respect to perturbations) the stability for the small perturbations is meant everywhere.

2. EQUATIONS OF MOTION

Equations of plane motion for an inextensional circular curve beam with no distributed load can be written as

$$F_n' + \gamma' F_t + G_n = 0 \tag{1}$$

$$F_t' - \gamma' F_n + G_t = 0 \tag{2}$$

$$F_n = M' = -EI\gamma'' \tag{3}$$

Here γ is an angle between a fixed (horizontal) axis x and the perpendicular n to the

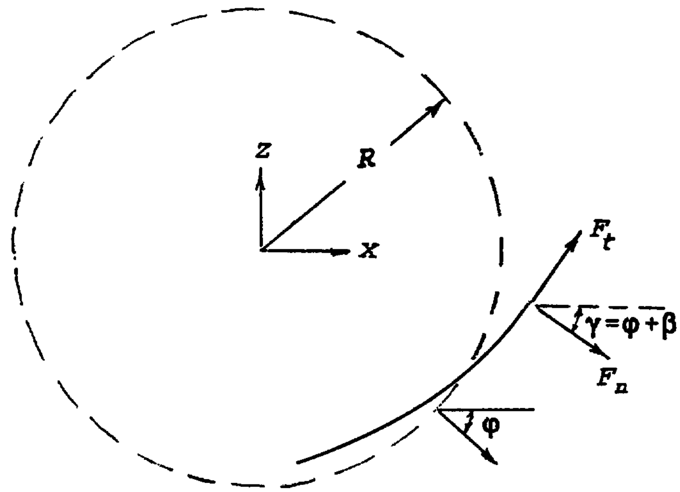

Fig. 1. Geometry and notations.

deformed beam surface (Fig. 1), F_n, F_t the shear and normal stress resultants referred to the distorted coordinate system (n, t), M the bending moment, EI the bending stiffness, G_n, G_t the normal and tangential components of inertia forces. A prime denotes the derivative with respect to the arc length $s = R\varphi$.

Components of inertial forces referred to a fixed axis (x, z) are

$$G_x = -\rho A\ddot{x}, \quad G_z = -\rho A\ddot{z}$$
$$x' = \sin\gamma, \quad z' = \cos\gamma \tag{4}$$

where ρ is mass density, A cross-section area and a dot indicates the time derivative. The inertial force components in the (n, t) coordinate system are

$$G_n = G_x\cos\gamma - G_z\sin\gamma, \quad G_t = G_x\sin\gamma + G_z\cos\gamma$$

Differentiating these equations and using (4) yields:

$$G_n' + \gamma' G_t + \rho A\ddot{\gamma} = 0, \quad G_t' - \gamma' G_n - \rho A\dot{\gamma}\dot{\gamma} = 0 \tag{5}$$

Equations (1)–(3) and (5), together with initial and boundary conditions, define stress resultants and rotation angle. When the latter is known (x, z) components of displacement (u_x, u_z) can be found from

$$u_x' = \sin\gamma - \sin\varphi, \quad u_z' = \cos\gamma - \cos\varphi \tag{6}$$

and (n, t) components from

$$w = u_x\cos\gamma - u_z\sin\gamma, \quad v = u_x\sin\gamma + u_z\cos\gamma \tag{7}$$

Another way of finding the radial and tangential components of the displacement is to integrate the system

$$w' + \gamma' v = \sin\beta, \quad v' - \gamma' w = 1 - \cos\beta, \quad \beta = \gamma - \varphi \tag{8}$$

which is easily obtainable by differentiating (7) and taking into account (6).

Introducing the dimensionless values

$$(S, T) = (R^3/EI)(G_n, G_t), \quad N = (R^2/EI)F_t, \quad \lambda^2 = \rho AR^4/EI$$

excluding the force F_n, and dropping all overbars, we rewrite equations of motion (1), (2) and (5) as follows

$$-\gamma''' + \gamma' N + S = 0 \tag{9}$$

$$N' + \gamma'\gamma'' + T = 0 \tag{10}$$

$$S' + \gamma' T + \lambda^2 \ddot{\gamma} = 0 \tag{11}$$

$$T' - \gamma' S - \lambda^2 \dot{\gamma}\dot{\gamma} = 0 \tag{12}$$

A prime now stands for the derivative with respect to φ.

If inertial forces (S, T) from the eqns (9), (10) are inserted into (11), (12), one could obtain eqns (35), (36) of Simmonds (1979), but we prefer to manipulate the system (9)–(12) directly.

3. EQUILIBRIUM OF AN ARCH UNDER CONCENTRATED LOAD

Consider the arch with guided ends loaded at one end by the static normal force $P/2$ (Fig. 2(a)). The end supports constrain rotation and tangential displacement but allow free radial displacement. Such a boundary conditions are realized at the ends of segment 1-1 of a ring loaded by the equally spaced forces P (Fig. 2(b)). At the equilibrium state $\gamma = S = T = 0$, the eqns (11), (12) are identically satisfied, and (10) yields

$$N' + \gamma'\gamma'' = 0$$

Upon integration we obtain the normal force

$$N = N_0 - (\gamma')^2/2 \tag{13}$$

where N_0 is an integration constant. The equation (9) now yields

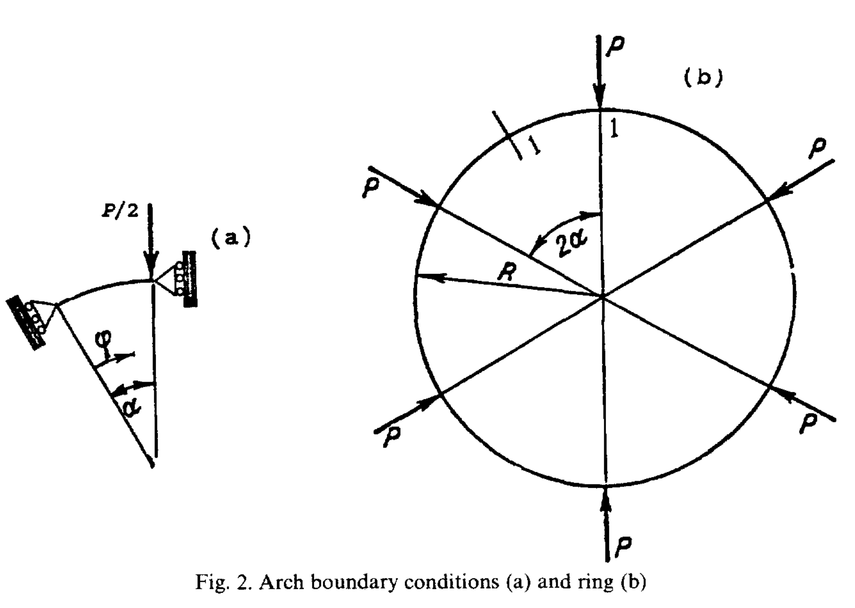

Fig. 2. Arch boundary conditions (a) and ring (b)

$$\Gamma'' - N_0\Gamma + \tfrac{1}{2}\Gamma^3 = 0, \quad \Gamma = \gamma' = \beta' + 1 \tag{14}$$

and its solution must satisfy boundary conditions

$$\beta(0) = \beta''(0) = 0 \tag{15}$$

$$\beta(\alpha) = 0, \quad -\beta''(\alpha) = P/2 \tag{16}$$

The solution of (14) can be expressed as

$$\Gamma = \Gamma_0 cn(h\varphi, k), \quad \Gamma' = -\Gamma_0 h sn(h\varphi, k)\, dn(h\varphi, k) \tag{17}$$

Here $cn(u,k)$, $sn(u,k)$, $dn(u,k)$ are Jacobi elliptic functions, k is modulus, and Γ_0, h are the integration constants. To calculate Jacobi functions we used their representation as an infinite trigonometric product, Gradstein and Ryzhik (1980), Davis (1962).

Inserting (17) into (14) and equating coefficient of $cn(h\varphi)$, $cn^3(h\varphi)$ to zero, we obtain

$$N_0 = (2k^2 - 1)h^2, \quad \Gamma_0^2 = 4k^2h^2 \tag{18}$$

Since both positive and negative values of Γ_0 are allowed, let

$$\Gamma_0 = 2kh\delta, \quad \delta = \pm 1 \tag{19}$$

The rotation by virtue of (14) and (17)

$$\beta(\varphi) = \int_0^{\varphi} [\Gamma_0 cn(h\varphi) - 1]\, d\varphi = -\varphi + 2\delta \arcsin(ksnh\varphi) \tag{20}$$

It is readily seen that conditions (15) are satisfied. Conditions (16) yield

$$ksn(u) = \delta \sin(\alpha/2), \quad u = h\alpha \tag{21}$$

$$P = 4kh^2\delta sn(u)\, dn(u) \tag{22}$$

Since Jacobi functions satisfy identity

$$dn^2(u) + k^2 sn^2(u) = 1$$

we find, using (21)

$$dn(u) = \sqrt{1 - k^2 sn^2(u)} = \cos(\alpha/2)$$

and (22) reduces to

$$P = 2(u/\alpha)^2 \sin\alpha \tag{23}$$

From (20) the rotation in the middle-span is

$$\beta_c = \beta(\alpha/2) = -(\alpha/2) + 2\delta \arcsin[ksn(u/2)] \tag{24}$$

For each given k the unknown u is to be found from (21) and from (24)–(23) we obtain a parametric relationship $P = P(\beta_c)$.

It is easily seen that for $1 > k \geqslant \sin(\alpha/2)$ equation (21) has an infinite number of real roots u_n, $n = 0, 1 \ldots$. Due to symmetry of $sn(u)$ about the points $u = K$ and $u = 3K$

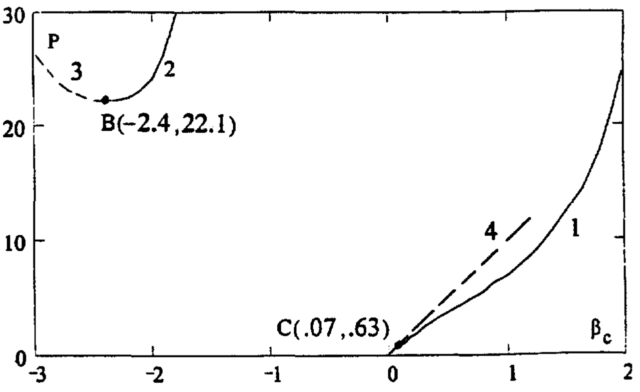

Fig. 3. Load–rotation curves for $\alpha = \pi/2$: (1) Primary path; (2) stable secondary path; (3) unstable secondary path; (4) linear solution.

$$u_1 - K = K - u_0, \quad u_1 = 2K - u_0 \quad \text{for } \delta = 1$$

$$u_1 - 3K = 3K - u_0, \quad u_1 = 6K - u_0 \quad \text{for } \delta = -1$$

where $K(k)$ is the complete elliptic integral of the first kind. All successive roots for $n > 1$ are found by simply adding period $4K$ to the preceding root so that only two roots u_0 for each k need to be found from (21).

The resulting load-rotation relationship for $\alpha = \pi/2$ is shown in Fig. 3, where the dashed straight line four represents the linear solution. Note, that there is an infinite number of bifurcation points at higher loads; only the lowest at $P = 22.1$ is shown in Fig. 3 (point B).

Some computation details should be mentioned here. The primary path 1 is formed by points (β_c, P), which correspond to the lowest roots u_0, u_1 of (21) with $\delta = 1$. For the lowest admissible modulus $k_0 = \sin(\alpha/2) = \sqrt{2}/2$ these roots are multiple and equal to

$$u_0 = u_1 = K(k_0) = 1.8541.$$

The corresponding point on the primary path is (0.36, 2.79). When modulus increases this root supplies up into two: $u_1 > u_0$. The points on the primary path corresponding to u_1 move away from the origin and the load tends to infinity when the modulus approaches unity.

Points corresponding to u_0 move in the direction of the origin $O(0,0)$ until the point $C(0.07, 0.63)$ is reached at $k = 1$. To cover a segment from point C to the origin one needs to take $k > 1$ and use for the computation an identity

$$ksn(u,k) = sn(ku, 1/k)$$

The segment CO is very close to the linear solution.

The secondary path 2, 3 in Fig. 3 is formed by two lowest roots for $\delta = -1$; two branches emanating from the next bifurcation point $B_1(-1.76, 66.4)$ correspond to u_2, u_3 for $\delta = 1$ and so on.

Formula (23) yields for a real u only positive (inward) forces. To include a case of an outward force it is necessary to replace u by iu. Since

$$sn(iu) = i\frac{sn(u)}{cn(u)}$$

we need at the same time replace k by ik to keep the left side of (21) real. As a result of these transformations function $cn(u)$ reduces to $1/dn(u)$ and in place of (17) the solution now is

$$\Gamma = \frac{\Gamma_0}{dn(h\varphi, k)}, \quad \Gamma' = \Gamma_0 hk^2 sn(h\varphi) cn(h\varphi)\, dn^{-2}(h\varphi) \tag{25}$$

Substitution in (14) yields

$$N_0 = (2-k^2)h^2, \quad \Gamma_0^2 = 4(1-k^2)h^2$$

and the end rotation

$$\beta(\alpha) = -\alpha + 2\delta\sqrt{1-k^2}\int_0^u \frac{dx}{dn(x)} = 0 \tag{26}$$

Since $dn(x) \geqslant \sqrt{1-k^2} > 0$ the integral is monotone increasing function of u and eqn (26) has only one real root for each k. Hence, for the outward forces the primary path is unique.

Usually in the problems of structural stability the portions of the load–displacement curve, where the increase of deformation requires a decrease in the applied loading, are unstable. By this criteria the dashed curve 3 in Fig. 3 is unstable. Beyond the load, corresponding to the bifurcation point B, both ascending paths are stable. If the load increases, the arch will follow the primary path, unless rotation is perturbed in the order of the distance between these two curves so that a jump to the secondary stable path becomes possible.

To explain this controversial behavior of the primary path consider the normal force $N(\varphi)$. According to (13) one obtains $N(0) = -h^2 < 0$ but at some point $\varphi = \varphi_0$ the normal force changes from a compression to tension. The coordinate φ_0 depends on the applied force as shown below for $\alpha = \pi/2$:

P	2.4	3.2	4.7	6.2	11.5	27.5
φ_0/α	1.0	0.87	0.58	0.45	0.26	0.15

Thus, the growth of the applied force causes the reduction in the portion of an arch subjected to the compressive normal force in such a way that the primary path remains stable.

The stability analysis is presented in the next section.

4. STABILITY ANALYSIS

To examine the stability of any state of the arch by the dynamic method we need to derive equations, describing small oscillations of the arch around this state. We introduce small disturbances $\bar{\gamma} = \bar{\beta}, \bar{F}, \bar{T}, \bar{Q}$ letting

$$\gamma = \bar{\gamma} + \gamma_s, \quad N = \bar{N} + N_s, \quad S = \bar{S} + S_s, \quad T = \bar{T} + T_s \tag{27}$$

where the subscript "s" denotes a state (solution), the stability of which is in question. Inserting (27) into (9)–(12), neglecting non-linear terms of disturbances, taking into account that solution "s" satisfies initial eqns (9)–(12), and dropping all overbars, we obtain

$$\begin{aligned} -\gamma''' + \gamma_s' N + N_s \gamma' + S &= 0 \\ N' + \gamma_s' \gamma'' + \gamma_s'' \gamma' + T &= 0 \\ S' + \gamma_s' T + T_s \gamma' + \lambda^2 \ddot{\gamma} &= 0 \\ T' - \gamma_s' S - S_s \gamma' - 2\lambda \dot{\gamma}_s \dot{\gamma} &= 0 \end{aligned} \tag{28}$$

If the solutions of (28) are bounded functions of time the state "s" is stable for the small disturbances.

At the equilibrium we had

$$\gamma_s' = \Gamma, \quad N_s = N_0 - \Gamma^2/2, \quad \dot{\gamma}_s = S_s = T_s = 0 \tag{29}$$

and hence

$$-\gamma''' + \Gamma N + (N_0 - \Gamma^2/2)\gamma' + S = 0 \tag{30}$$

$$N' + \Gamma\gamma'' + \Gamma'\gamma' + T = 0 \tag{31}$$

$$S' + \Gamma T + \lambda^2\ddot{\gamma} = 0 \tag{32}$$

$$T' - \Gamma S = 0 \tag{33}$$

Function γ must satisfy homogenous boundary conditions corresponding to (15), (16) or

$$\gamma(0) = \gamma(\alpha) = \gamma''(0) = \gamma''(\alpha) = 0 \tag{34}$$

These boundary conditions are satisfied assuming solution in the form:

$$\gamma = \exp(i\omega t)\sum_{n=1} \gamma_n \sin\mu_n\varphi, \quad T = \exp(i\omega t)\sum_{n=1} T_n \sin\mu_n\varphi$$

$$N = \exp(i\omega t)\sum_{n=0} N_n \cos\mu_n\varphi, \quad S = \exp(i\omega t)\sum_{n=0} S_n \cos\mu_n\varphi \tag{35}$$

where $i = \sqrt{-1}$, $\mu_n = n\pi/\alpha$. Using Galerkin's procedure, we insert (35) into (30)–(33), multiply (31) and (32) by $\sin(\mu_j\varphi)\,d\varphi, j = 1, 2\ldots$, (30) and (33) by $\cos(\mu_j\varphi)\,d\varphi, j = 0, 1\ldots$, and integrate over the arch length. This yields the following linear homogeneous system:

$$0.5\alpha(\mu_j N_j - T_j) + \mu_j \Sigma\mu_n B_{jn}\gamma_n = 0$$

$$0.5\alpha(\mu_j S_n + \lambda^2\omega^2\gamma_j) - \Sigma C_{jn} T_n = 0$$

$$0.5\alpha[(\mu_j^3 + N_0\mu_j)\gamma_j + S_j] + \Sigma(B_{jn}N_n - 0.5\mu_n A_{jn}\gamma_n) = 0$$

$$0.5\alpha\mu_j T_j - \Sigma B_{jn} S_n = 0$$

$$\alpha S_0 + \Sigma(B_{0n}N_n - 0.5\mu_n A_{0n}\gamma_n) = 0, \quad \Sigma B_{0n}S_n = 0 \tag{36}$$

Here all summations are from $n = 0$ to infinity. In the first four equations $j = 1, 2\ldots$; the last two ones correspond to $j = 0$. Coefficients in (36) are:

$$A_{jn} = \int \Gamma^2 \cos(\mu_j\varphi)\cos(\mu_n\varphi)\,d\varphi$$

$$B_{jn} = \int \Gamma \cos(\mu_j\varphi)\cos(\mu_n\varphi)\,d\varphi$$

$$C_{jn} = \int \Gamma \sin(\mu_n\varphi)\sin(\mu_n\varphi)\,d\varphi$$

where all integrals are from 0–α.

Equating determinant of (36) to zero, one can find squared frequency of vibration ω^2. Positiveness of this value ensures stability. Retaining in (36) the constant terms and coefficients of two first harmonics it was found that the primary path and the ascending branch of secondary path, emanating from the first bifurcation point B, are stable while the descending branch is unstable. Both paths emanating from the second bifurcation point are unstable.

5. CONCLUSION

By the exact solution of the non-linear governing equations it is shown that the circular guided arch loaded at one end by the concentrated force remains stable as the load tends to infinity. There is an infinite number of bifurcation points, not located on the primary path. Two branches of the secondary path, one stable and one unstable, emanate from the first bifurcation point.

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